

Automorphism groups of Beauville surfaces

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Abstract

A Beauville surface of unmixed type is a complex algebraic surface which is the quotient of the product of two curves of genus at least 2 by a finite group G acting freely on the product, where G preserves the two curves and their quotients by G are isomorphic to the projective line, ramified over three points. We show that the automorphism group A of such a surface has an abelian normal subgroup I isomorphic to the centre of G , induced by pairs of elements of G acting compatibly on the curves (a result obtained independently by Fuertes and González-Diez). Results of Singerman on inclusions between triangle groups imply that A/I is isomorphic to a subgroup of the wreath product $S_3 \wr S_2$, so A is a finite solvable group. Using constructions based on Lucchini's work on generators of special linear groups, we show that every finite abelian group can arise as I , even if one restricts the index $|A : I|$ to the extreme values 1 or 72.

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1 Introduction

A Beauville surface \mathcal{S} is a complex algebraic surface which is rigid and is isogenous to a higher product, that is, it has the form $(\mathcal{C}_1 \times \mathcal{C}_2)/G$, where \mathcal{C}_1 and \mathcal{C}_2 are complex algebraic curves of genus at least 2, and G is a finite group acting freely on their product. Here we restrict attention to Beauville surfaces of unmixed type, where G preserves the factors \mathcal{C}_i , and rigidity means that \mathcal{C}_i/G is isomorphic to the projective line $\mathbb{P}^1(\mathbb{C})$, with the covering $\mathcal{C}_i \rightarrow \mathbb{P}^1(\mathbb{C})$ ramified over three points. (This implies that each \mathcal{C}_i carries a regular dessin, in the sense of Grothendieck's theory of *dessins d'enfants* [20], so by Belyi's Theorem [4] \mathcal{S} is defined over an algebraic number field.) The first example, with \mathcal{C}_1 and \mathcal{C}_2 Fermat curves, was introduced by Beauville in [3], and subsequently these surfaces have been studied by geometers such as Bauer, Catanese and Grunewald [1, 2, 6]. Much recent research has concentrated on the question of which groups G , and in particular which simple groups,

can arise in this context [11, 12, 13, 15, 16, 17, 21]. Here we take a different point of view, studying the automorphism group A of \mathcal{S} , and how it is related to G .

The automorphisms of a Beauville surface \mathcal{S} are of two types, which we will call *direct* or *indirect* as they lift to automorphisms of $\mathcal{C}_1 \times \mathcal{C}_2$ which either preserve or transpose the two factors \mathcal{C}_i . The direct automorphisms form a subgroup $A^0 = \text{Aut}^0 \mathcal{S}$ of index at most 2 in the group $A = \text{Aut} \mathcal{S}$ of all automorphisms of \mathcal{S} . Indirect automorphisms exist if and only if the two curves \mathcal{C}_i are isomorphic, with the actions of G on them transposed by an automorphism of G . We will concentrate mainly on the group A^0 . We show that this has an abelian normal subgroup $I = \text{Inn} \mathcal{S}$ isomorphic to the centre $Z(G)$ of G ; its elements, which we call the *inner automorphisms* of \mathcal{S} , are induced by pairs of elements of G acting compatibly on the curves \mathcal{C}_i . In most cases $A^0 = I$, but in some cases either curve \mathcal{C}_i may have additional automorphisms corresponding to non-identity permutations of the three ramification points, and these may induce what we call *outer automorphisms* of \mathcal{S} . The ramification condition on each \mathcal{C}_i is equivalent to G arising as a quotient of a hyperbolic triangle group by a normal surface subgroup uniformising \mathcal{C}_i . Using results of Singerman [27] on inclusions between triangle groups, we show that the *outer automorphism group* $\text{Out} \mathcal{S} = A/I$ is isomorphic to a subgroup of the wreath product $S_3 \wr S_2$, a semidirect product of $S_3 \times S_3$ by S_2 , so that A is a finite solvable group. A result of Lucchini [25] on generators of special linear groups allows us to show that every finite abelian group is isomorphic to $\text{Inn} \mathcal{S}$ for some Beauville surface \mathcal{S} ; moreover, we show that \mathcal{S} can be chosen here so that $\text{Out} \mathcal{S}$ is as large or as small as possible, namely isomorphic to $S_3 \wr S_2$ or the trivial group. We also give further examples with $\text{Out} \mathcal{S}$ lying between these two extremes, for instance showing that every finite generalised dihedral group can arise as $\text{Aut} \mathcal{S}$ for some \mathcal{S} .

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2 Beauville surfaces and structures

A *Beauville surface* (of unmixed type) is a compact complex surface \mathcal{S} such that

- (a) \mathcal{S} is isogenous to a higher product, that is, $\mathcal{S} \cong (\mathcal{C}_1 \times \mathcal{C}_2)/G$ where \mathcal{C}_1 and \mathcal{C}_2 are algebraic curves of genus at least 2 and G is a finite group acting faithfully on \mathcal{C}_1 and \mathcal{C}_2 by holomorphic transformations in such a way that it acts freely on $\mathcal{C}_1 \times \mathcal{C}_2$;
- (b) each \mathcal{C}_i/G is isomorphic to the projective line $\mathbb{P}^1(\mathbb{C})$, and the covering $\mathcal{C}_i \rightarrow \mathcal{C}_i/G$ is ramified over three points.

(We will not consider Beauville surfaces of mixed type, where G contains elements which transpose the two curves \mathcal{C}_i .) Condition (b) is equivalent to each curve \mathcal{C}_i admitting a regular *dessin* in the sense of Grothendieck's theory of *dessins d'enfants* [20], or equivalently an orientably regular hypermap [23], with G acting as the orientation-preserving automorphism group. By Belyi's Theorem [4], this implies that the curves \mathcal{C}_i and the coverings $\mathcal{C}_i \rightarrow \mathbb{P}^1(\mathbb{C})$ are defined over algebraic number fields, and hence the same applies to \mathcal{S} .

A finite group G arises in this way if and only if it has generating triples a_i, b_i and c_i for $i = 1, 2$, of orders l_i, m_i and n_i , such that

- (1) $a_i b_i c_i = 1$ for each $i = 1, 2$,
- (2) $l_i^{-1} + m_i^{-1} + n_i^{-1} < 1$ for each $i = 1, 2$, and
- (3) no non-identity power of a_1, b_1 or c_1 is conjugate in G to a power of a_2, b_2 or c_2 .

We will call such a pair of triples (a_i, b_i, c_i) a *Beauville structure* for G . Property (1) is equivalent to G being a smooth quotient Δ_i/K_i of a triangle group $\Delta_i = \Delta(l_i, m_i, n_i)$ by a normal surface subgroup K_i uniformising \mathcal{C}_i , with a_i, b_i and c_i the local monodromy permutations for the covering $\mathcal{C}_i \rightarrow \mathcal{C}_i/G$ at the three ramification points; we call l_i, m_i and n_i the *elliptic periods* of Δ_i . Property (2) is equivalent to each \mathcal{C}_i having genus at least 2, so that Δ_i acts on the hyperbolic plane \mathbb{H} , with $\mathcal{C}_i \cong \mathbb{H}/K_i$, and property (3) is equivalent to G acting freely on $\mathcal{C}_1 \times \mathcal{C}_2$.

To show that a pair such as a_1 and a_2 satisfy property (3) it is sufficient to verify that, for each prime p dividing both l_1 and l_2 , the element $a_1^{l_1/p}$ of order p is not conjugate to any of the elements $a_2^{kl_2/p}$ where $k = 1, 2, \dots, p-1$; in particular, if l_1 is prime then it is sufficient to verify this for a_1 . Similar remarks apply to every other pair chosen from the two triples, so in particular property (3) is satisfied if $l_1 m_1 n_1$ is coprime to $l_2 m_2 n_2$.

A *triple* in a group G will always mean an ordered triple (a, b, c) of elements of G such that $abc = 1$; it is a *generating triple* if a, b and c (and hence any two of them) generate G , and it has *type* (l, m, n) if these *periods* l, m and n are the orders of a, b and c . Two types are *equivalent* if they differ by a permutation of their periods, so that they define the same triangle groups, but with different generating triples. We will say that a Beauville structure, in the notation above, has *type* $(l_1, m_1, n_1; l_2, m_2, n_2)$; here equivalence allows us to permute or transpose the two multisets $\{l_i, m_i, n_i\}$.

3 Automorphism groups

Let $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ be a Beauville surface, as described in Section 2. Any automorphism α of \mathcal{S} lifts to an automorphism $\bar{\alpha}$ of $\mathcal{C}_1 \times \mathcal{C}_2$, and this is of one of the two following types, where $A_i := \text{Aut } \mathcal{C}_i$ for $i = 1, 2$ (see [6]):

- 1. $\bar{\alpha} = (\alpha_1, \alpha_2) : p = (p_1, p_2) \mapsto (p_1 \alpha_1, p_2 \alpha_2)$ where $\alpha_i \in A_i$ for $i = 1, 2$, or
- 2. $\bar{\alpha} = (\phi_1, \phi_2) : p = (p_1, p_2) \mapsto (p_2 \phi_2, p_1 \phi_1)$ where $\phi_1 : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and $\phi_2 : \mathcal{C}_2 \rightarrow \mathcal{C}_1$ are isomorphisms.

Let us call automorphisms α and $\bar{\alpha}$ *direct* or *indirect*, as $\bar{\alpha}$ is of type 1 or 2 respectively. The direct automorphisms form a subgroup of $\text{Aut } (\mathcal{C}_1 \times \mathcal{C}_2)$ isomorphic to $A_1 \times A_2$. This is the whole of $\text{Aut } (\mathcal{C}_1 \times \mathcal{C}_2)$ unless $\mathcal{C}_1 \cong \mathcal{C}_2$ ($\cong \mathcal{C}$, say), in which case $\text{Aut } (\mathcal{C}_1 \times \mathcal{C}_2)$ is isomorphic to the wreath product $\text{Aut } \mathcal{C} \wr S_2$, a semidirect product of $A_1 \times A_2 \cong (\text{Aut } \mathcal{C})^2$ by a complement S_2 transposing the direct factors.

3.1 Direct automorphisms

A direct automorphism $(\alpha_1, \alpha_2) \in A_1 \times A_2$ of $\mathcal{C}_1 \times \mathcal{C}_2$ induces an automorphism of \mathcal{S} if and only if, whenever $p = (p_1, p_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ and $q = (q_1, q_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ are equivalent under G , then so are their images under (α_1, α_2) . More explicitly, we require that if $p_i g = q_i$ for $g \in G$ and $i = 1, 2$, then there is some $h \in G$ such that $(p_i \alpha_i)h = q_i \alpha_i$ for $i = 1, 2$. If it exists, then such an element h is unique, and independent of the points p_i , since a generic point has trivial stabiliser, so we can write this condition as $\alpha_i^{-1} g \alpha_i = h \in G$ for $i = 1, 2$. We require this for all $g \in G$, so α_i must be an element of the normaliser $N_i := N_{A_i}(G)$ of G in A_i for $i = 1, 2$, with the induced automorphisms $\beta_i : g \mapsto h$ of G satisfying $\beta_1 = \beta_2$, i.e. the natural homomorphisms $\theta_i : N_i \rightarrow \text{Aut } G$ induce

$$\theta = (\theta_1, \theta_2) : N_1 \times N_2 \rightarrow \text{Aut } G \times \text{Aut } G$$

sending (α_1, α_2) to an element of the diagonal subgroup E of $\text{Aut } G \times \text{Aut } G$. Thus (α_1, α_2) induces an automorphism of \mathcal{S} if and only if it lies in $N := \theta^{-1}(E)$.

Such a pair (α_1, α_2) acts trivially on \mathcal{S} if and only if there is some $g \in G$ such that $p_i \alpha_i g = p_i$ for all $p \in \mathcal{C}_i$ ($i = 1, 2$), i.e. $\alpha_i \in G$ for $i = 1, 2$. Thus the kernel of the action of N on \mathcal{S} is the diagonal subgroup D of $G \times G \leq N_1 \times N_2 \leq A_1 \times A_2$, so the group $A^0 = \text{Aut}^0 \mathcal{S}$ of direct automorphisms of \mathcal{S} has the form

$$A^0 \cong N/D. \tag{1}$$

In all cases, $G \leq N_i$ and the restriction $\theta_i|_G : G \rightarrow \text{Aut } G$ is simply the action of G on itself by conjugation, with kernel equal to the centre $Z := Z(G)$, so N contains a normal subgroup

$$M = N \cap (G \times G) = \{(g_1, g_2) \in G \times G \mid g_1 g_2^{-1} \in Z\} = D \times Z,$$

where the direct factor Z can be taken to be the centre of either direct factor of $G \times G$. Hence A^0 contains a normal subgroup $I := \text{Inn } \mathcal{S} \cong M/D \cong Z$, consisting of the *inner automorphisms* of \mathcal{S} , those induced by compatible pairs of elements of G acting on the curves \mathcal{C}_i , or equivalently by elements of Z acting naturally on one curve and fixing the other. Since I is isomorphic to the centre Z of G , it is finite and abelian. We will call the quotient group $A^0/I \cong N/M$ the *direct outer automorphism group* $\text{Out}^0 \mathcal{S}$ of \mathcal{S} .

In most cases $G = A_i$, so $G = N_i$ and $N = M = D \times Z$; then (1) implies that in such cases we have

$$A^0 = I \cong Z. \tag{2}$$

The only possible exceptions to (2) are where G is a proper subgroup of N_i for some i , so that Δ_i is a proper normal subgroup of a Fuchsian group $\tilde{\Delta}_i$, with $\tilde{\Delta}_i/K_i \cong N_i$. As shown by Singerman in [27], since $\tilde{\Delta}_i$ contains a triangle group it must also be a triangle group. He showed that the only possibilities for a proper normal inclusion $\Delta \triangleleft \tilde{\Delta}$ of one hyperbolic triangle group $\Delta = \Delta_i$ in another triangle group $\tilde{\Delta} = \tilde{\Delta}_i$ are (up to permutations of the periods) of the form

$$(a) \Delta(s, s, t) \triangleleft \Delta(2, s, 2t), \quad (b) \Delta(t, t, t) \triangleleft \Delta(3, 3, t), \quad (c) \Delta(t, t, t) \triangleleft \Delta(2, 3, 2t)$$

for some integers s and t , with $\tilde{\Delta}/\Delta \cong C_2, C_3$ or S_3 respectively. In all three cases, at least two of the three periods of Δ are equal, so we have:

Proposition 3.1 *If a Beauville structure on a group G has type $(l_1, m_1, n_1; l_2, m_2, n_2)$, and for each i the periods l_i, m_i and n_i are mutually distinct, then the direct automorphism group $\text{Aut}^0 \mathcal{S}$ of the corresponding Beauville surface \mathcal{S} is isomorphic to the centre $Z(G)$ of G . \square*

In each of the exceptional cases (a), (b) and (c), let u, v and w be the canonical elliptic generators of $\tilde{\Delta}$, with $uvw = 1$. In case (a) we can take Δ to have elliptic generators $v^u = uvu$, v and w^2 , of orders s, s and t , with $v^u.v.w^2 = (uv)^2w^2 = 1$; then conjugation by u induces an automorphism of Δ transposing v^u and v . In case (b) we can take Δ to have generators w , $w^{u^2} = uwu^{-1}$ and $w^u = u^{-1}wu$, all of order t , with $w.w^{u^2}.w^u = (wu)^3 = v^{-3} = 1$; conjugation by u induces a 3-cycle on these generators. The inclusion in case (c) is a composition of two normal inclusions

$$\Delta(t, t, t) \triangleleft \Delta(3, 3, t) \triangleleft \Delta(2, 3, 2t),$$

of types (b) and (a), with generators v^u, v and w^2 for $\Delta(3, 3, t)$ and hence $w^2, (w^2)^{(v^u)^2}$ and $(w^2)^{v^u}$ for $\Delta(t, t, t)$; in this case

$$w^2.(w^2)^{(v^u)^2}.(w^2)^{v^u} = (w^2uvu)^3 = (wu)^3 = v^{-3} = 1,$$

and the element v^u induces a 3-cycle on these generators. In each of cases (a), (b) and (c), if a normal subgroup K of Δ is also normal in $\tilde{\Delta}$, then the generators α of $N = \tilde{\Delta}/K$ corresponding to the elliptic generators of $\tilde{\Delta}$ must induce automorphisms β of $G = \Delta/K$, acting as above on the generators of G corresponding to those of Δ . Conversely, if a quotient $G = \Delta/K$ of Δ has automorphisms β acting in this way on its generators, then by forming appropriate semidirect products with the groups $\langle \beta \rangle$ (twice in case (c)), we see that K is normal in $\tilde{\Delta}$ and G is a normal subgroup of index 2, 3 or 6 in $N = \tilde{\Delta}/K$.

If G admits a Beauville structure then by applying the above arguments to G as a quotient of the appropriate triangle groups Δ_i , one can determine the groups N_i . By considering the automorphisms of G induced by each N_i one can then determine N and hence A^0 , using (1).

For each $i = 1, 2$ we have $N_i/G \cong \tilde{\Delta}_i/\Delta_i$, isomorphic to a subgroup of S_3 , so $A^0/I \cong N/M$ is isomorphic to a subgroup of $S_3 \times S_3$. Since this group has order 36, and is solvable of derived length 2, we have the following generalisation of Proposition 3.1, most of which has also been obtained by Fuertes and González-Díez in [14]:

Proposition 3.2 *If \mathcal{S} is a Beauville surface, obtained from a Beauville structure on a group G , then the direct automorphism group $\text{Aut}^0 \mathcal{S}$ of \mathcal{S} has an abelian normal subgroup $I \cong Z(G)$ with quotient group $\text{Out}^0 \mathcal{S}$ isomorphic to a subgroup of $S_3 \times S_3$. In particular $\text{Aut}^0 \mathcal{S}$ is solvable, of derived length at most 3, and it has order dividing $36|Z(G)|$. \square*

See Section 4.5 and Theorem 5.6, and also [14, §5], for instances in which $|\text{Out}^0 \mathcal{S}|$ attains the upper bound of 36.

There is a natural interpretation of the embedding of $\text{Out}^0 \mathcal{S}$ in $S_3 \times S_3$. The covering $\mathcal{C}_i \rightarrow \mathcal{C}_i/G \cong \mathbb{H}/\Delta_i \cong \mathbb{P}^1(\mathbb{C})$ is a Belyĭ function, that is, a non-constant meromorphic function unbranched outside three points; it is convenient to apply a Möbius transformation of $\mathbb{P}^1(\mathbb{C})$ so that these points are 0, 1 and ∞ . They represent the orbits of G on \mathcal{C}_i with nontrivial stabilisers, namely cyclic groups of orders l_i, m_i and n_i , so in the language of orbifolds they are cone-points of these orders. Since Δ_i is normal in $\tilde{\Delta}_i$, there is an action of $\tilde{\Delta}_i$ as a group of automorphisms of $\mathbb{P}^1(\mathbb{C})$ leaving invariant the set $B = \{0, 1, \infty\}$. The kernel of this action is Δ_i , and there is a corresponding action of $N_i \cong \tilde{\Delta}_i/K_i$, with kernel $G \cong \Delta_i/K$, inducing a subgroup of S_3 on B . We therefore obtain a product action of $N_1 \times N_2$, with kernel $G \times G$, on B^2 ; this induces a subgroup of $S_3 \times S_3$, preserving the equivalence relations \equiv_i on B^2 defined by pairs having the same i th component, for $i = 1, 2$. Restricting this action to N , with kernel M , we obtain a faithful action of $N/M \cong \text{Out}^0 \mathcal{S}$ on B^2 as a subgroup of $S_3 \times S_3$.

3.2 Splitting properties

In the normal inclusions (a) and (b), the triangle group $\tilde{\Delta}$ is a semidirect product of Δ by C_2 or C_3 respectively: in case (a) we can take $\langle u \rangle$ as a complement for Δ in $\tilde{\Delta}$, and in case (b) we can take $\langle u \rangle$ or $\langle v \rangle$. It follows that if N_i contains G with index 2 or 3 then it is a split extension of G , with the image of u or v generating the complement. However, in case (c) $\tilde{\Delta}$ does not split over Δ : a finite subgroup of a hyperbolic (or euclidean) triangle group must be cyclic, so $\tilde{\Delta}$ has no subgroups isomorphic to the quotient group $\tilde{\Delta}/\Delta \cong S_3$.

We have $A^0/I \cong N/M$, and it follows from the above argument that if this group has order 2 or 3 then A^0 and N split over I and M . If A^0/I has order 4 then involutions in $N_i \setminus G$ ($i = 1, 2$) commute and generate a Klein four-group V_4 complementing M in N , so A^0 is a semidirect product of I by V_4 . A similar argument applies if $|A^0/I| = 9$, giving a complement $C_3 \times C_3$.

These splitting properties imply that not every group which satisfies the conclusions of Proposition 3.2 can arise as the direct automorphism group of a Beauville surface.

Example 3.1 For each integer $e \geq 2$ the generalised quaternion group

$$Q = \langle g, h \mid g^{2^e} = 1, g^h = g^{-1}, h^2 = g^{2^{e-1}} \rangle$$

of order 2^{e+1} is an extension of a cyclic normal subgroup $\langle g \rangle$ of order 2^e by C_2 , so it satisfies the conclusions of Proposition 3.2. It is nonabelian, so if it arises as the direct automorphism group A^0 of a Beauville surface then $A^0 > I$, and hence $|A^0 : I| = 2$ or 4 since this index is a power of 2 dividing 36. Any normal subgroup of index 2 or 4 in Q has an abelian quotient, so it contains the commutator $[g, h] = g^{-2}$; it therefore contains $g^{2^{e-1}}$, which is the only element of order 2 in Q , so Q cannot split over such a normal subgroup. It follows that no Beauville surface can satisfy $A^0 \cong Q$.

3.3 Indirect automorphisms

Any indirect automorphism of $\mathcal{C}_1 \times \mathcal{C}_2$ has the form

$$t : \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{C}_1 \times \mathcal{C}_2, (p_1, p_2) \mapsto (p_2\phi_2, p_1\phi_1), \quad (3)$$

where $\phi_1 : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and $\phi_2 : \mathcal{C}_2 \rightarrow \mathcal{C}_1$ are isomorphisms of curves (in which case there are $|\text{Aut } \mathcal{C}_1| = |\text{Aut } \mathcal{C}_2|$ possibilities for each). Equivalently, we can identify \mathcal{C}_1 and \mathcal{C}_2 via ϕ_1 , call the resulting curve \mathcal{C} , and define

$$t : \mathcal{C}^2 \rightarrow \mathcal{C}^2, (p_1, p_2) \mapsto (p_2\phi, p_1)$$

where $\phi = \phi_2$ is now an automorphism of \mathcal{C} .

In order to allow for the possibility (in fact, the necessity) of G acting in different ways on the two factors, it is useful when considering indirect automorphisms to regard G as acting on each \mathcal{C}_i by means of a faithful representation $\rho_i : G \rightarrow G_i \leq A_i$. Thus, when we write $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$, we are really factoring out the action of the image of the diagonal subgroup of $G \times G$ under the product representation $\rho_1 \times \rho_2 : G \times G \rightarrow A_1 \times A_2$.

Any indirect automorphism τ of \mathcal{S} must be induced by some t of the form (3). If they exist, such automorphisms of \mathcal{S} are all obtained by composing a specific indirect automorphism τ (in either order) with an arbitrary direct automorphism of \mathcal{S} . Now t induces an automorphism of \mathcal{S} if and only if, whenever some $g \in G$ sends (p_1, p_2) to (q_1, q_2) in $\mathcal{C}_1 \times \mathcal{C}_2$, there exists some $h \in G$ sending $(p_1, p_2)t = (p_2\phi_2, p_1\phi_1)$ to $(q_1, q_2)t = (q_2\phi_2, q_1\phi_1)$, that is, $p_2\phi_2(h\rho_1) = p_2(g\rho_2)\phi_2$ and $p_1\phi_1(h\rho_2) = p_1(g\rho_1)\phi_1$. If such an element h exists then it is unique and is independent of p_1 and p_2 , so we require $\phi_2(h\rho_1) = (g\rho_2)\phi_2$ and $\phi_1(h\rho_2) = (g\rho_1)\phi_1$. The mapping $g \mapsto h$, if it exists, is an automorphism ζ of G ; we then require $(g\zeta)\rho_1 = \phi_2^{-1}(g\rho_2)\phi_2$ and $(g\zeta)\rho_2 = \phi_1^{-1}(g\rho_1)\phi_1$ for all $g \in G$, so that ϕ_2 induces an equivalence between the representations ρ_2 and $\zeta \circ \rho_1$ of G , while ϕ_1 induces an equivalence between ρ_1 and $\zeta \circ \rho_2$. The two representations ρ_1 and ρ_2 cannot be equivalent, for otherwise condition (3) of a Beauville structure would not hold. It follows that ζ transposes the two equivalence classes of representations ρ_i ; in particular, ζ cannot be an inner automorphism of G . Conversely, if G has an automorphism transposing these two classes, then the isomorphisms ϕ_i realising these equivalences give an indirect automorphism of \mathcal{S} . Thus we have proved:

Proposition 3.3 *A Beauville surface $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ has an indirect automorphism if and only if $\mathcal{C}_1 \cong \mathcal{C}_2$ and G has an automorphism ζ transposing the equivalence classes of its representations on \mathcal{C}_1 and \mathcal{C}_2 . \square*

Corollary 3.4 *If a Beauville surface $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ has an indirect automorphism, then the corresponding Beauville structure for G must consist of two triples of equivalent types.*

Proof. The automorphism ζ of G must preserve the orders of the stabilisers of points in the representations of G on the two curves. \square

We will initially use Corollary 3.4 to show that various Beauville surfaces do not possess indirect automorphisms. Later, in Section 6, we will consider indirect automorphisms in more detail, and give examples of Beauville surfaces with such automorphisms.

4 Examples of direct automorphism groups

This section contains some specific examples of automorphism groups which illustrate the general results proved in Section 3.

4.1 Examples with trivial automorphism groups

In [13], Fuertes and González-Diez have shown that the symmetric groups $G = S_n$ admit Beauville structures for all $n \geq 5$. They give examples of Beauville structures of types $(2, n-2, n-3; 2, n, n-1)$ and $(2, 4(n-6), n-2; 2, n-1, n)$ respectively for even and odd $n > 10$. These types satisfy the conditions of Proposition 3.1, and the groups S_n have trivial centres, so the corresponding Beauville surfaces have $A^0 = 1$. In each case the two triples have inequivalent types, so it follows from Corollary 3.4 that $A = 1$ also.

4.2 Examples with automorphism group C_2

Let G be the simple group $L_2(p) = SL_2(p)/\{\pm I\}$ for some prime $p \equiv 1 \pmod{24}$. Since $p \equiv 1 \pmod{4}$ there is some u in the field \mathbb{F}_p such that $u^2 = -1$. Let

$$a_1 = \pm \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad b_1 = \pm \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \quad \text{and} \quad c_1 = (a_1 b_1)^{-1} = \pm \begin{pmatrix} 1 & -u \\ -u & 0 \end{pmatrix},$$

so that a_1 and b_1 have order p , and c_1 has order 3. The maximal subgroups of G have been classified by Dickson [9, Ch. XII], and of these, only the stabilisers of points in the projective line $\mathbb{P}^1(p)$ have order divisible by p ; since a_1 and b_1 have no common fixed points no such subgroup contains them both, so they generate G . Now let

$$a_2 = \pm \begin{pmatrix} 0 & v \\ -v^{-1} & w \end{pmatrix}, \quad b_2 = \pm \begin{pmatrix} w & -v^{-1} \\ v & 0 \end{pmatrix} \quad \text{and} \quad c_2 = (a_2 b_2)^{-1} = \pm \begin{pmatrix} v^{-2} & 0 \\ w(v^{-1} - v) & v^2 \end{pmatrix},$$

where we choose the trace $\pm w$ of a_2 and b_2 so that they have order $(p+1)/2$. The order of c_2 , one quarter that of v in \mathbb{F}_p^* and thus dividing $(p-1)/4$, can be chosen to be coprime to 3 by a suitable choice of v , e.g. taking $v^2 = u$ so that c_2 has order 2. By [9, Ch. XII] the only maximal subgroups of G containing elements of order $(p+1)/2$ are dihedral groups of order $p+1$, and a_2 and b_2 cannot both be contained in such a subgroup since they do not commute. It follows that a_2 and b_2 generate G . Since the orders of a_1, b_1 and c_1 are coprime to those of a_2, b_2 and c_2 , and both triples are hyperbolic, they define a Beauville structure on G . The element

$$g = \pm \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \in G,$$

acting by conjugation, induces an automorphism $\beta_1 = \beta_2$ of order 2 of G , transposing the generators a_i and b_i for each $i = 1, 2$. Thus G has index 2 in $N_i = A_i$ for each i , with the inclusions of triangle groups as in case (a). Since $Z(G) = 1$ it follows that N is a semidirect product of $D \cong G$ by $\langle \alpha_i \rangle \cong C_2$ (in fact $N \cong G \times C_2$, since α_i induces an inner automorphism β_i of G), and hence $A^0 \cong C_2$. The two triples have inequivalent types, so $A = A^0$.

4.3 Examples based on Fermat curves

Let $G = C_t \times C_t$, with two generating triples of type (t, t, t) , as in Beauville's original example [3], where $t = 5$, and in Catanese's generalisation [6], where t is coprime to 6. Each C_i is isomorphic to the Fermat curve \mathcal{F}_t of genus $(t-1)(t-2)/2$ given by $x^t + y^t + z^t = 0$. The triangle group $\Delta_i = \Delta(t, t, t)$ is a normal subgroup of index 6 in $\tilde{\Delta}_i = \Delta(2, 3, 2t)$, as in case (c) in Section 3.1. Since K_i is the commutator subgroup Δ'_i of Δ_i , it is a characteristic subgroup of Δ_i and hence normal in $\tilde{\Delta}_i$, so $A_i = N_i \cong \tilde{\Delta}_i/K_i$ is an extension of $G \cong \Delta_i/K_i$ by $\tilde{\Delta}_i/\Delta_i \cong S_3$ for $i = 1, 2$. This extension splits, with the normal subgroup given by multiplying the homogeneous coordinates x, y and z by powers of $e^{2\pi i/t}$, and a complement given by permuting them. Since G is abelian, we have $G \times G \leq \ker \theta \leq N$. Thus $M = G \times G$, and A^0 contains a normal subgroup $I = M/D = (G \times G)/D \cong G$.

Whether A^0 properly contains I depends on the choice of generating triples a_i, b_i, c_i defining the Beauville structure on G : specifically, as shown in Section 3.1, we need to decide whether G has automorphisms inducing transpositions or 3-cycles on both of them. For simplicity, let us take t to be a prime $p \geq 5$, so that two generating triples define a Beauville structure if and only if their images in the projective line $\mathbb{P}^1(p)$, formed by the 1-dimensional subgroups of G , are disjoint. Given a generating triple a_1, b_1, c_1 for G , the 3-cycle (a_1, b_1, c_1) extends to a unique automorphism β_1 of G , which decomposes $G \setminus \{1\}$ into $(p^2 - 1)/3$ cycles (a_2, b_2, c_2) of length 3. Now $p - 1$ of these are scalar multiples of (a_1, b_1, c_1) , and if $p \equiv 2 \pmod{3}$ then each of the remaining 3-cycles (a_2, b_2, c_2) generates G , satisfies $a_2 b_2 c_2 = 1$, and has a disjoint image from that of (a_1, b_1, c_1) in $\mathbb{P}^1(p)$, so it gives a Beauville structure on G ; the automorphism β_2 of G it induces coincides with β_1 , giving an element β of order 3 in N . The situation is similar if $p \equiv 1 \pmod{3}$, except that in order to generate G the triple a_2, b_2, c_2 must now avoid the two 1-dimensional $\langle \beta_1 \rangle$ -invariant subgroups of G . However, if in either case we choose a_2, b_2, c_2 not to form a 3-cycle of β_1 , as is possible provided $p > 5$, then there is no element of order 3 in N .

By contrast, transposing elements of generating triples never induces automorphisms of the Beauville surface \mathcal{S} . Since G is abelian, a transposition of two elements of a triple induces an automorphism of G fixing the third. Such an automorphism has a 1-dimensional subgroup of fixed points in G , so there cannot be an element of order 2 in A^0/I : the two fixed elements (one from each triple) would be multiples of each other, contradicting condition (3) for a Beauville structure.

These arguments show that if the triples are chosen to be invariant under the same automorphism $\beta = \beta_1 = \beta_2$ of order 3, then N is a semidirect product of $G \times G$ by $\langle \beta \rangle \cong C_3$, so A^0 is a semidirect product of $I \cong G = C_p \times C_p$ by $\langle \beta \rangle$, with β acting on G as above. Any other choice of triples (possible if $p > 5$) gives $N = G \times G$ and $A^0 = I \cong G = C_p \times C_p$.

We have shown that if \mathcal{S} is constructed from the Fermat curve \mathcal{F}_t , where t is a prime $p \geq 5$, then $|\text{Out}^0 \mathcal{S}| = 1$ or 3. One can extend this result to all t coprime to 6 by using the natural epimorphism $C_{p^e} \rightarrow C_p$ to deal with prime powers, and for general t using the direct product decomposition of C_t based on the prime power factorisation of t . See [19] for details, including an enumeration of these Beauville surfaces extending asymptotic estimates by Garion and Penegini [16]. We will consider indirect automorphisms in Section 6.

4.4 A useful construction

The following lemma will be useful for the next example, and also for later constructions.

Lemma 4.1 *Let G be a finite group which is a smooth quotient of $\tilde{\Delta} := \Delta(2, 3, n)$ and has no subgroups of index 2. Then G is also a smooth quotient Δ/K of $\Delta := \Delta(t, t, t)$, where $t = n/2$ or n as n is even or odd, with $\tilde{\Delta}/K \cong G \times S_3$. If $t > 3$ then the surface group K has normaliser $N(K) = \tilde{\Delta}$ in $PSL_2(\mathbb{R})$.*

Proof. First let $n = 2t$ be even, so there is a normal surface subgroup L of $\tilde{\Delta} = \Delta(2, 3, 2t)$ with $\tilde{\Delta}/L \cong G$. Singerman's normal inclusion (c) shows that $\tilde{\Delta}$ has a normal subgroup $\Delta := \Delta(t, t, t)$ with $\tilde{\Delta}/\Delta \cong S_3$. Now $\Delta L/\Delta$ is a normal subgroup of $\tilde{\Delta}/\Delta$, corresponding to a normal subgroup of S_3 , which must be 1, A_3 or S_3 . In the first two cases $\tilde{\Delta}/\Delta L$ has a quotient isomorphic to $S_3/A_3 \cong C_2$, and hence so does $\tilde{\Delta}/L$, against our hypotheses about G . It follows that $\Delta L = \tilde{\Delta}$, so if we define $K = \Delta \cap L$ then $\Delta/K \cong \tilde{\Delta}/L \cong G$ and $\tilde{\Delta}/K \cong G \times S_3$. Since L is a surface group, so is its subgroup K , so G is a smooth quotient of Δ . If $t > 3$ then the normaliser $N(K)$ of K in $PSL_2(\mathbb{R})$ contains $\tilde{\Delta}$; these are Fuchsian groups, and since Singerman [27] has shown that $\tilde{\Delta}$ is maximal among Fuchsian groups, we have $N(K) = \tilde{\Delta}$.

If $n = t$ is odd we can regard G as a quotient of $\tilde{\Delta} = \Delta(2, 3, 2t)$ via the natural epimorphism $\tilde{\Delta} \rightarrow \Delta(2, 3, n)$. Although G is no longer a smooth quotient, the only torsion elements in the kernel L are elliptic elements of order 2, conjugate to w^t where w is the canonical generator of $\tilde{\Delta}$ of order $2t$. However the image of w in $\tilde{\Delta}/\Delta$ has order 2, so only even powers of w lie in Δ ; thus $K = \Delta \cap L$ is torsion-free and is therefore a surface group. The rest of the proof is as before. \square

Corollary 4.2 *If a finite group G satisfies the conditions of Lemma 4.1 for two mutually coprime values t_1 and t_2 of t , with each $t_i > 3$, then G admits a Beauville structure of type $(t_1, t_1, t_1; t_2, t_2, t_2)$ with $A = A^0 = I \times S_3 \times S_3 \cong Z(G) \times S_3 \times S_3$.*

Proof. The existence of the Beauville structure follows immediately from Lemma 4.1. We have $N_i \cong G \times S_3$ for $i = 1, 2$, with G acting on itself by inner automorphisms, and S_3 centralising G . The arguments in Section 3.2 then give $N = M \times S_3 \times S_2 = D \times Z \times S_3 \times S_3$, so $A^0 = I \times S_3 \times S_3 \cong Z(G) \times S_3 \times S_3$. The two triples have inequivalent types, so by Corollary 3.4 there are no indirect automorphisms. \square

As we shall show, this allows the construction of examples in which $|A^0 : I|$ attains its upper bound of 36.

4.5 Examples with automorphism group $S_3 \times S_3$

As an application of Corollary 4.2, let $G = L_2(p)$ for a prime $p > 11$, and let

$$a_i = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b_i = \pm \begin{pmatrix} u & v \\ w & 1-u \end{pmatrix} \quad \text{and} \quad c_i = (a_i b_i)^{-1} = \pm \begin{pmatrix} -v & u-1 \\ u & w \end{pmatrix},$$

where $u(1-u) - vw = 1$. Then a_i and b_i have traces 0 and ± 1 , so they have orders 2 and 3. For $i = 1$ let us take $u = 0$, so $w = -1/v$, and choose v to have order $p-1$ in \mathbb{F}_p^* , so that c_1 , having trace $\pm(v + v^{-1})$, has order $(p-1)/2 > 5$. For $i = 2$ let us take $u = 1, v = -1$ and $w = 1$, so that c_2 has order p . For each i it follows from Dickson's classification of the maximal subgroups of G in [9, Ch. XII] that the triple a_i, b_i, c_i generates G . Thus G is a smooth quotient of $\Delta(2, 3, n)$ with $n = (p-1)/2$ and with $n = p$, so it satisfies the conditions of Lemma 4.1 with $t = t_1 = (p-1)/4$ or $(p-1)/2$ as $p \equiv \pm 1 \pmod{4}$, and also with $t = t_2 = p$. Since t_1 and t_2 are coprime, and $Z(G) = 1$, Corollaries 3.4 and 4.2 show that G has a Beauville structure of type $(t_1, t_1, t_1; t_2, t_2, t_2)$ with $A = A^0 \cong S_3 \times S_3$.

One can construct many similar examples, with $G = L_2(q)$ for suitable prime powers q , by using Macbeath's results [26] on generating triples for these groups. One can also use this method to construct examples where G is an alternating group A_n . Conder [7] has shown that A_n is a quotient of $\Delta(2, 3, 7)$ (i.e. a Hurwitz group) for all sufficiently large n , and Everitt [10], as part of a more general result on Fuchsian groups, has extended this to all hyperbolic triangle groups, such as $\Delta(2, 3, t)$ for $t \geq 7$. One can ensure that these quotients are smooth (most easily by taking t to be prime), so the preceding arguments show that A_n has Beauville structures with $A = A^0 \cong S_3 \times S_3$ for all sufficiently large n . Using a direct construction, Fuertes and González-Diez [14, §5] have given an explicit example of such a Beauville structure for A_{15} .

5 Realising abelian groups

Proposition 3.1 shows that the inner automorphism group $I = \text{Inn } \mathcal{S}$ of a Beauville surface \mathcal{S} is a finite abelian group, isomorphic to the centre of the corresponding finite group G . The main aim of this section is to show that every finite abelian group can arise as $\text{Inn } \mathcal{S}$ for some Beauville surface \mathcal{S} , even if extra restrictions are imposed on $\text{Out}^0 \mathcal{S}$. The general strategy is as follows. Every finite abelian group H is a direct product of cyclic groups. Special linear groups $SL_d(q)$ have cyclic centres, of all possible orders, so by taking G to be a direct product of suitable special linear groups we can arrange that $Z(G)$ is isomorphic to H . Using results of Lucchini [25] we can ensure that these special linear groups are quotients of certain triangle groups, and hence, using a lemma which we shall shortly prove, so is their product G . In this way we can construct G to have a Beauville structure, with $I \cong Z(G) \cong H$, and with $\text{Out}^0 \mathcal{S}$ satisfying various other conditions.

Lucchini's result [25] is as follows:

Proposition 5.1 *For each integer $t \geq 7$ there exists an integer d_t such that $SL_d(q)$ is a quotient of $\Delta(2, 3, t)$ for all $d \geq d_t$ and all prime powers q . \square*

The centre of $SL_d(q)$ is a cyclic group of order $\gcd(d, q-1)$, consisting of the matrices λI_d where $\lambda^d = 1$ in \mathbb{F}_q . In order to apply Proposition 5.1 we need the following lemma:

Lemma 5.2 *Given any integer $m \geq 1$, there exist infinitely many integers $d \geq 2$, for each of which there are infinitely many prime powers q with $Z(SL_d(q)) \cong C_m$.*

Proof. Let $d = rm$ where r is any integer coprime to $2m$. Since m and r are coprime, it follows from the Chinese Remainder Theorem and Dirichlet's Theorem on primes in arithmetic progressions that there are infinitely many primes q satisfying $q \equiv 1 \pmod{m}$ and $q \equiv -1 \pmod{r}$. In such cases m divides both d and $q - 1$, so it divides their highest common factor h . Since $d/m = r$, which is an odd divisor of $q + 1$ and hence coprime to $q - 1$, it follows that $m = h$. Thus $Z(SL_d(q)) \cong C_m$. \square

Our aim is to construct a group G , with $Z(G)$ isomorphic to an arbitrary finite abelian group H , by taking a direct product of various groups G_j of type $SL_d(q)$, one for each cyclic direct factor of H . We need to show that if each of the groups G_j has a Beauville structure, then so has G . In order to achieve this, let us define two groups to be *mutually orthogonal* if only the trivial group is a quotient of both of them. This concept was introduced in [5], and the following result generalises Corollary 18(a) of that paper:

Lemma 5.3 *Let K_1, \dots, K_k be normal subgroups of a group Γ such that the quotient groups $G_j = \Gamma/K_j$ are mutually orthogonal, and let $K = K_1 \cap \dots \cap K_k$. Then $\Gamma/K \cong G_1 \times \dots \times G_k$.*

Proof. Using induction on k , it is sufficient to consider the case $k = 2$. In this case $K_1 K_2 = \Gamma$, for otherwise $\Gamma/K_1 K_2$ is a nontrivial common quotient of G_1 and G_2 . Then $\Gamma/K = K_1/K \times K_2/K \cong \Gamma/K_2 \times \Gamma/K_1 \cong G_2 \times G_1 \cong G_1 \times G_2$. \square

Corollary 5.4 *Let G_1, \dots, G_k be mutually orthogonal finite groups. If each G_j admits a Beauville structure, then so does $G := G_1 \times \dots \times G_k$.*

Proof. Let triples (a_{ij}, b_{ij}, c_{ij}) for $i = 1, 2$ define a Beauville structure on G_j for each $j = 1, \dots, k$. Define elements a_i, b_i, c_i of G by $a_i = (a_{i1}, \dots, a_{ik})$, and so on. Then $a_i b_i c_i = 1$ for each i , and Beauville condition (2) is satisfied since it is satisfied in at least one (in fact every) direct factor G_j . If some power a_1^r of a_1 is conjugate in G to a power a_2^s of a_2 , then the same applies to a_{1j}^r and a_{2j}^s in each G_j , so $a_{1j}^r = 1$ for each j and hence $a_1^r = 1$; a similar argument applies to any other pair chosen from a_1, b_1, c_1 and a_2, b_2, c_2 . For each i , by mapping the two canonical generators of the free group $\Gamma = F_2$ to a_{ij} and b_{ij} , we represent each G_j as a quotient Γ/K_j of Γ . Lemma 5.2 then shows that G is also a quotient of Γ , generated by a_i and b_i . Thus both triples generate G , giving a Beauville structure on this group. \square

Note that if the Beauville structure on each G_j has type $(l_{1j}, m_{1j}, n_{1j}; l_{2j}, m_{2j}, n_{2j})$, then that on G has type $(l_1, m_1, n_1; l_2, m_2, n_2)$ where $l_i = \text{lcm}(l_{i1}, \dots, l_{ik})$, etc.

Lemma 5.5 *Distinct groups $SL_d(q)$ for $d \geq 2$ are mutually orthogonal, except that $SL_2(4)$ and $SL_2(5)$ have a common quotient $SL_2(4) \cong A_5 \cong L_2(5)$, and $SL_3(2)$ and $SL_2(7)$ have a common quotient $SL_3(2) \cong L_2(7)$.*

Proof. Apart from $SL_2(2)$ and $SL_2(3)$, these groups are all perfect. Their only simple quotients are the groups $L_d(q)$, and these are mutually non-isomorphic apart from the isomorphisms $L_2(4) \cong L_2(5)$ and $L_3(2) \cong L_2(7)$ (see [8], for instance). The solvable groups $SL_2(2)$ and $SL_2(3)$ have only C_2 and C_3 respectively as simple quotients, so this argument extends to them. \square

Theorem 5.6 *Each finite abelian group H is isomorphic to the inner automorphism group $\text{Inn } \mathcal{S}$ of a Beauville surface \mathcal{S} with $\text{Aut } \mathcal{S} = \text{Aut}^0 \mathcal{S} \cong H \times S_3 \times S_3$.*

Proof. We will apply Corollary 4.2 to a suitable group G with $Z(G) \cong H$. If H is the identity group we can use the example in Section 4.5, so we may assume that $H \cong C_{m_1} \times \cdots \times C_{m_k}$ for integers $m_j \geq 2$. Let us choose two distinct primes $t_1, t_2 \geq 7$. By Proposition 5.1, if $d \geq \max\{d_{t_1}, d_{t_2}\}$ then $SL_d(q)$ is a quotient of $\tilde{\Delta}_i = \Delta(2, 3, t_i)$ for all q and for each $i = 1, 2$. Since 2, 3 and t_i are primes it must be a smooth quotient. By Lemmas 5.2 and 5.5 we can therefore choose mutually orthogonal groups G_1, \dots, G_k of type $SL_d(q)$ which are smooth quotients of $\tilde{\Delta}_i$ for $i = 1, 2$ and have centres $Z(G_j) \cong C_{m_j}$ for $j = 1, \dots, k$. The group $G = G_1 \times \cdots \times G_k$ has centre $Z(G) \cong H$, and by Lemma 5.3 it is also a smooth quotient of each $\tilde{\Delta}_i$. Having no subgroups of index 2, G satisfies the hypotheses of Lemma 4.1, so by Corollary 4.2 it has a Beauville structure of type $(t_1, t_1, t_1; t_2, t_2, t_2)$ with $I \cong H$ and $A = A^0 \cong H \times S_3 \times S_3$. \square

Remarks. 1. One can obtain the slightly weaker result that every finite abelian group is isomorphic to $\text{Inn } \mathcal{S}$ for some Beauville surface \mathcal{S} by combining Corollary 5.4 with recent results of Fairbairn, Magaard and Parker [12] and of Garion, Larson and Lubotzky [17] which show that, with just finitely many exceptions, the groups $SL_d(q)$ for $d \geq 2$ all admit Beauville structures. This avoids the use of Lemma 5.2, since there is now no requirement that d should be sufficiently large.

2. The construction used to prove Theorem 5.6 can be adapted to produce other Beauville surfaces, still realising H as their inner automorphism group, but with smaller direct outer automorphism groups. For instance, let G_0 be a symmetric group S_{11} , with the Beauville structure constructed by Fuertes and González-Diez in [13] and described in Section 4.1, having type $(2, 20, 9; 2, 10, 11)$ and only the identity automorphism. If G_1, \dots, G_k are as in the proof of Theorem 5.6, then since G_0 is mutually orthogonal to them, Corollary 5.4 implies that the group $G = G_0 \times G_1 \times \cdots \times G_k$ has a Beauville structure. Since $Z(G_0) = 1$, this structure has $I \cong Z(G) \cong H$ as before. However, the type of this new structure consists of two distinct triples, each with no repetitions, so we have $A = A^0 = I$. This proves:

Theorem 5.7 *Every finite abelian group is isomorphic to the automorphism group $\text{Aut } \mathcal{S}$ of some Beauville surface \mathcal{S} .* \square

Proposition 3.2 shows that, for any Beauville surface \mathcal{S} , the group $\text{Out}^0 \mathcal{S}$ is isomorphic to a subgroup of $S_3 \times S_3$. Theorems 5.6 and 5.7 show that the two extreme cases can arise, where $\text{Out}^0 \mathcal{S}$ is the whole group or the trivial group, with $\text{Inn } \mathcal{S}$ isomorphic to an arbitrary finite abelian group H . Similar constructions, using suitable normal inclusions of Δ_i in $\tilde{\Delta}_i$, show that other intermediate subgroups of $S_3 \times S_3$ can also arise, specifically from direct automorphism groups of the form $A^0 \cong H \times H_1 \times H_2$, where each $\tilde{\Delta}_i/\Delta_i \cong H_i \leq S_3$.

Example 5.1 Let us take $\Delta_1 = \Delta(7, 7, 7)$ and $\tilde{\Delta}_1 = \Delta(2, 3, 14)$ (as in the proof of Theorem 5.6, with $t_1 = 7$); by Proposition 5.1, if $d \geq d_8$ then $SL_d(q)$ is a quotient of

$\Delta_2 := \Delta(2, 3, 8)$ for all q , necessarily smooth since $\Delta(2, 3, 4) (\cong S_4)$ has only $SL_2(2)$ ($\cong S_3$) as a quotient of this type. Since 7 is coprime to 2, 3 and 8, the construction used for Theorem 5.6 yields Beauville structures of type $(7, 7, 7; 2, 3, 8)$ with $I \cong H$; the normal inclusion $\Delta_1 \triangleleft \tilde{\Delta}_1$ gives $H_1 \cong S_3$ as before, while the maximality of the triangle group Δ_2 (see [27]) implies that $\tilde{\Delta}_2 = \Delta_2$ and hence $H_2 = 1$. The corresponding Beauville surfaces \mathcal{S} therefore have $\text{Aut}^0 \mathcal{S} \cong H \times S_3$, so $\text{Out}^0 \mathcal{S} \cong S_3$.

Example 5.2 A similar construction, taking $\tilde{\Delta}_2 = \Delta(2, 3, 8)$ and Δ_2 its subgroup $\Delta(3, 3, 4)$ of index 2, gives Beauville structures of type $(7, 7, 7; 3, 3, 4)$ with $\text{Aut}^0 \mathcal{S} \cong H \times S_3 \times C_2$ and thus $\text{Out}^0 \mathcal{S} \cong S_3 \times C_2$.

In the proofs of Theorem 5.6 and 5.7, and in the above examples, I is in the centre of A^0 , but Section 4.3 shows that this is not always the case, at least when $I \cong C_p \times C_p$ for some prime p . Here we give further examples, with I isomorphic to other groups.

Example 5.3 Let S be a non-identity finite group with a generating triple (x, y, z) of type (l, m, n) . Let $G = S \times S \times S$, and let G^* be the subgroup of G generated by

$$a_1 = (x, y, z), \quad b_1 = (y, z, x) \quad \text{and} \quad c_1 = (z, x, y),$$

so that $a_1 b_1 c_1 = 1$ since $xyz = yzx = zxy = 1$. Suppose that l, m and n are mutually coprime (for instance, S could be a Hurwitz group, with $(l, m, n) = (2, 3, 7)$). Then G^* contains $a_1^{mn} = (x^{mn}, 1, 1)$ and hence contains $(x, 1, 1)$ since x is a power of x^{mn} . Similar arguments show that G^* contains generators for all three direct factors of G . Thus $G^* = G$, and hence G is a quotient of $\Delta := \Delta(t, t, t)$ where $t = lmn$.

Since $x^{-1} z^{-1} y^{-1} = 1$, essentially the same argument shows that the elements

$$a_2 = (x^{-1}, z^{-1}, y^{-1}), \quad b_2 = (z^{-1}, y^{-1}, x^{-1}) \quad \text{and} \quad c_2 = (y^{-1}, x^{-1}, z^{-1})$$

of order t generate G and satisfy $a_2 b_2 c_2 = 1$. No non-identity power of y can be conjugate in S to a power of z (since they have coprime orders), so by considering their second coordinates we see that the same applies to a_1 and a_2 in G . In fact this applies to any pair of elements chosen from the first and the second of these two triples. Since $t > 3$ they therefore form a Beauville structure in G .

The automorphism $\theta : (g_1, g_2, g_3) \mapsto (g_2, g_3, g_1)$ of G has order 3 and permutes a_i, b_i and c_i cyclically for each i , so the corresponding extension \tilde{G} of G by $\langle \theta \rangle$ (the wreath product $S \wr C_3$) is a quotient of $\tilde{\Delta} = \Delta(3, 3, t)$, with the same kernel K as G . Thus $|N_i : G|$ divisible by 3 for $i = 1, 2$.

To show that $|N_1 : G| = 3$ it is sufficient to show that G has no automorphism transposing a_1 and b_1 . Such an automorphism would transpose $a_1^{mn} = (x^{mn}, 1, 1)$ and $b_1^{mn} = (1, 1, x^{mn})$, and hence $(x, 1, 1)$ and $(1, 1, x)$; similarly it would transpose a_1^{ln} and b_1^{ln} , and hence $(1, y, 1)$ and $(y, 1, 1)$; since $(1, 1, x)$ and $(1, y, 1)$ commute, so must $(x, 1, 1)$ and $(y, 1, 1)$, which is impossible since their product $(z^{-1}, 1, 1)$ has order n coprime to lm . A similar argument shows that $|N_2 : G| = 3$.

Since N_1 and N_2 induce the same group of automorphisms of G (both acting as \tilde{G}), it follows that the direct automorphism group A^0 of the corresponding Beauville surface is

a semidirect product of $I \cong Z(G) = Z(S) \times Z(S) \times Z(S)$ by $\langle \theta \rangle \cong C_3$, with θ permuting the three direct factors $Z(S)$ in a 3-cycle, so that $A^0 \cong Z(S) \wr C_3$. As in the proof of Theorem 5.6, by applying Proposition 5.1 to $\Delta(2, 3, n)$ with $n (\geq 7)$ coprime to 6 one can choose S so that $Z(S)$ is isomorphic to any given finite abelian group.

6 Beauville surfaces with indirect automorphisms

We now return to the situation in Section 3.3, where \mathcal{S} has an indirect automorphism $\tau : (p_1, p_2) \mapsto (p_2\phi_2, p_1\phi_1)$. Here ϕ_1 and ϕ_2 are isomorphisms $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ and $\mathcal{C}_2 \rightarrow \mathcal{C}_1$, and the representations ρ_i of G on the curves \mathcal{C}_i satisfy $\zeta \circ \rho_1 = \phi_2^{-1} \rho_2 \phi_2$ and $\zeta \circ \rho_2 = \phi_1^{-1} \rho_1 \phi_1$ for some $\zeta \in \text{Aut } G$. Any other indirect automorphism τ' is obtained by composing τ with a direct automorphism (α_1, α_2) ; these automorphisms α_i of the curves \mathcal{C}_i can be absorbed into new isomorphisms ϕ'_i , leaving ζ unchanged, so ζ is independent of the choice of $\tau \in A \setminus A^0$.

6.1 Normality of the inner automorphism group

A direct automorphism of \mathcal{S} has the form $\alpha = (\alpha_1, \alpha_2) \in N_1 \times N_2$, with α_1 and α_2 inducing the same automorphism of G by conjugation (see Section 3.1). A simple calculation shows that the action of τ by conjugation on A^0 is given by $\alpha^\tau = (\alpha_2^{\phi_2}, \alpha_1^{\phi_1})$. Now α is an inner automorphism of \mathcal{S} if and only if each $\alpha_i = g_i \rho_i$ for some $g_i \in G$ with $z := g_1 g_2^{-1} \in Z$, in which case $\alpha^\tau = ((g_2 \rho_2)^{\phi_2}, (g_1 \rho_1)^{\phi_1}) = ((g_2 \zeta) \rho_1, (g_1 \zeta) \rho_2)$ with $(g_2 \zeta)(g_1 \zeta)^{-1} = z^{-1} \zeta \in Z$. Thus the indirect automorphisms normalise I , acting by conjugation as the automorphism

$$z \mapsto z^\tau = z^{-1} \zeta = (z \zeta)^{-1}, \quad (4)$$

where we identify I with Z by means of the isomorphism $\alpha \mapsto z$. Since I is normal in A^0 , it follows that I is normal in A .

As explained at the end of Section 3.3, there is an action of $\text{Out}^0 \mathcal{S} = A^0/I$ as a subgroup of $S_3 \times S_3$, permuting the set B^2 where $B = \{0, 1, \infty\}$, and preserving the relations \equiv_i on B^2 of having the same i th component. Any indirect automorphism τ acts on B^2 by transposing these two equivalence relations, so it induces an element of the wreath product $S_3 \wr S_2$, the largest group of permutations of B^2 preserving $\{\equiv_1, \equiv_2\}$. This proves the following analogue of Proposition 3.2:

Proposition 6.1 *If \mathcal{S} is a Beauville surface, obtained from a Beauville structure on a group G , then the automorphism group $\text{Aut } \mathcal{S}$ of \mathcal{S} has an abelian normal subgroup $I \cong Z(G)$ with quotient group $\text{Out } \mathcal{S}$ isomorphic to a subgroup of $S_3 \wr S_2$. In particular $\text{Aut } \mathcal{S}$ is solvable, of derived length at most 4, and it has order dividing $72 |Z(G)|$. \square*

6.2 Triangle group inclusions

If \mathcal{S} is a Beauville surface in which \mathcal{C}_1 and \mathcal{C}_2 are isomorphic, then we may regard them as a single curve \mathcal{C} , uniformised by the same surface group K . This must be a normal

subgroup of hyperbolic triangle groups Δ_1 and Δ_2 , with each $\Delta_i/K \cong G$. Since each Δ_i normalises K it is contained in the normaliser $N(K)$ of K in $PSL_2(\mathbb{R})$, and as shown by Singerman [27], this must also be a triangle group Δ^* .

If \mathcal{S} is to have indirect automorphisms, then Δ_1 and Δ_2 must be of the same type. Now it has been shown by Gironde and Wolfart in [18, Theorem 13] that, except in one special case, if two hyperbolic triangle groups Δ_1 and Δ_2 of the same type are contained in another triangle group Δ^* , then they are conjugate in Δ^* . In our situation, where $\Delta^* = N(K)$, we may without loss of generality conjugate one of them by an element of Δ^* and thus take them to be the same triangle group Δ . Then the two actions of Δ on \mathcal{C} as Δ_1 and Δ_2 , having the same kernel K , must (up to equivalence) differ by an automorphism of G ; if they are to form a Beauville structure, this must be an outer automorphism.

The exceptional case arises when Δ_1 and Δ_2 both have type $(n, 2n, 2n)$ for some integer $n \geq 3$, and are distinct subgroups of index 2 in $\Delta^* = \Delta(2, 2n, 2n)$, namely the normal closures in Δ^* of its two canonical generators of order $2n$; these are clearly not conjugate in Δ^* , though they are conjugate in a triangle group $\Delta(2, 4, 2n)$ which contains Δ^* as a subgroup of index 2. (These inclusions are all of Singerman's type (a) for various choices of the parameters s and t .) If this situation is to yield a Beauville surface with indirect automorphisms, then not only must Δ_1/K and Δ_2/K be isomorphic (to G), but the corresponding two actions of G on \mathcal{C} must differ only by automorphisms of \mathcal{C} and G , as shown in Section 6.1. However, this implies that Δ_1 and Δ_2 must be conjugate in $N(K) = \Delta^*$, which is false, so this case cannot lead to Beauville surfaces with indirect automorphisms. We will therefore assume from now on that $\Delta_1 = \Delta_2 = \Delta$.

6.3 Examples of indirect automorphisms

Example 6.1 The examples of Beauville surfaces \mathcal{S} in [3, 6], based on the Fermat curves \mathcal{F}_t , have $\Delta = \Delta(t, t, t)$, $K = \Delta'$, $\Delta^* = \Delta(2, 3, 2t)$ and $G = C_t \times C_t$, with t coprime to 6; here the inclusion of Δ in Δ^* is of Singerman's type (c). In Section 4.3 we showed that if t is a prime $p \geq 5$ then $\text{Out}^0 \mathcal{S}$ has order 1 or 3; it follows that $\text{Out} \mathcal{S}$ has order 1, 2, 3 or 6 (see [19] for the extension to all t coprime to 6). Here we give examples of all four cases.

Two generating triples (a_i, b_i, c_i) in $G = C_p \times C_p$ yield a Beauville surface \mathcal{S} if and only if their images in the projective line $\mathbb{P}^1(p)$ form disjoint sets Σ_i . Each A_i is a semidirect product of the abelian group $G_i = G\rho_i \cong G$ by S_3 , so equivalence of representations ρ_i corresponds to permuting the elements in a generating triple. By Proposition 3.3, if \mathcal{S} has an indirect automorphism then some element of $PGL_2(p)$ transposes Σ_1 and Σ_2 .

We can regard G as a 2-dimensional vector space over \mathbb{F}_p , and without loss of generality we can choose coordinates so that the first generating triple is $a_1 = (1, 0)$, $b_1 = (0, 1)$, $c_1 = (-1, -1)$. The matrix

$$B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \in GL_2(p),$$

acting on the left of column vectors, defines an automorphism β of G inducing a 3-cycle (a_1, b_1, c_1) on this triple. If we let each $(i, j) \neq (0, 0)$ correspond to the point $i/j \in \mathbb{P}^1(p)$,

then this generating triple corresponds to the triple $\Sigma_1 = \{\infty, 0, 1\}$ in $\mathbb{P}^1(p)$. If we choose the second generating triple a_2, b_2, c_2 to form a 3-cycle of β , inducing a triple $\Sigma_2 \subset \mathbb{P}^1(p)$ disjoint from Σ_1 , we obtain a Beauville surface \mathcal{S} with $\text{Out}^0 \mathcal{S} \cong C_3$ (see Section 4.3). Then \mathcal{S} also has indirect automorphisms if and only if there is an automorphism ζ of G transposing these two generating triples. In this case, $\text{Out} \mathcal{S}$ has order 6 and is therefore isomorphic to C_6 or S_3 as ζ centralises or inverts β .

Apart from $-I$, which does not lead to a Beauville structure, the involutions in $GL_2(p)$ which commute with B are the matrices of the form

$$Z = \begin{pmatrix} u & -2u \\ 2u & -u \end{pmatrix}$$

where $3u^2 = -1$. By quadratic reciprocity, \mathbb{F}_p contains such elements u if and only if $p \equiv 1 \pmod{3}$. The automorphism ζ induced by Z transposes a_1, b_1, c_1 with the generating triple $a_2 = (u, 2u)$, $b_2 = (-2u, -u)$, $c_2 = (u, -u)$. This corresponds to a triple $\Sigma_2 = \{1/2, 2, -1\} \subset \mathbb{P}^1(p)$ disjoint from Σ_1 , so these generating triples a_i, b_i, c_i form a Beauville structure in G with $\text{Out} \mathcal{S} \cong C_6$.

Similarly, the involutions in $GL_2(p)$ inverting B are the matrices

$$Z = \begin{pmatrix} u & v \\ u+v & -u \end{pmatrix}$$

where $u^2 + uv + v^2 = 1$. Such a matrix induces an automorphism ζ of G which permutes the 3-cycles of β , reversing their cyclic order. It transposes the generating triple a_1, b_1, c_1 with the triple $a_2 = (u, u+v)$, $b_2 = (v, -u)$, $c_2 = (-u-v, -v)$, these triples forming a Beauville structure if and only if $u, v, u+v \neq 0$. We then find that $\text{Out} \mathcal{S} \cong \langle B, Z \rangle \cong S_3$. Suitable values of u and v exist for all primes $p > 7$: in the projective plane $\mathbb{P}^2(p)$, the conic $u^2 + uv + v^2 = w^2$ has $p+1$ points, of which either two or none are on the line at infinity $w = 0$ as $p \equiv 1$ or $-1 \pmod{p}$, so provided $p > 7$ there are points on the affine curve $u^2 + uv + v^2 = 1$ besides the six points $(\pm 1, 0)$, $(0, \pm 1)$ and $(\pm 1, \mp 1)$ we need to avoid.

If we form a Beauville structure by choosing for the second triple a 3-cycle of β which is not of one of the above two types, then we obtain a Beauville surface \mathcal{S} with $\text{Out} \mathcal{S} = \text{Out}^0 \mathcal{S} \cong C_3$.

Example 6.2 Further examples are given by the groups

$$G = \langle g, h \mid g^{p^e} = h^{p^e} = 1, h^g = h^q \rangle,$$

where p is an odd prime and $q = 1 + p^f$ with $f = 1, 2, \dots, e$. These groups arose in [22] in connection with regular embeddings of complete bipartite graphs, and various associated regular dessins were studied in [24]. Such a group G is a semidirect product of a normal subgroup $\langle h \rangle$ by $\langle g \rangle$, both cyclic of order $n = p^e$, so each element of G has the unique form $g^i h^j$ where $i, j \in \mathbb{Z}_n$. (This group is a direct product if and only if $e = f$, in which case we have a Fermat curve, as in Section 4.3 and Example 6.1.) As shown in [22, Corollary 10], G has exponent n , the elements of this order being those of $G \setminus \Phi$, where Φ is the Frattini

subgroup consisting of those elements with $i \equiv j \equiv 0 \pmod{p}$. By [22, Corollary 11], two cyclic subgroups of order n in G have trivial intersection if and only if their images in $G/\Phi \cong C_p \times C_p$ also do. It follows from this that two triples form a Beauville structure in G if and only if their images form a Beauville structure in G/Φ . We therefore obtain Beauville structures in G provided $p \geq 5$. Let us choose $a_1 = gh$, $b_1 = g^{-4}h^{-2}$ and $c_1 = (a_1b_1)^{-1}$, so the corresponding values of $i/j \in \mathbb{P}^1(p)$ are 1, 2 and 3. The above presentation shows that G has an automorphism ζ of order 2 fixing g and inverting h . Applying ζ to the triple a_1, b_1, c_1 , we obtain a second triple a_2, b_2, c_2 with $i/j = -1, -2$ and -3 , values which are disjoint from those for the first triple provided $p \geq 7$. In such cases, Proposition 3.3 shows that the corresponding Beauville surface has an indirect automorphism.

It is shown in [22, §4] that the group G in Example 6.2 has centre

$$Z(G) = \langle g^{p^{e-f}}, h^{p^{e-f}} \rangle \cong C_{p^f} \times C_{p^f},$$

so the corresponding surface has non-identity inner automorphisms. For later use we will now give a class of examples with an indirect automorphism, but only the identity inner automorphism.

Example 6.3 Let $P = \langle g, h \mid g^p = h^p = 1, gh = hg \rangle \cong C_p \times C_p$, for a prime $p \geq 13$. The triples

$$\bar{a}_1 = g^2h, \bar{b}_1 = g^{-6}h^{-2}, \bar{c}_1 = g^4h \quad \text{and} \quad \bar{a}_2 = gh^2, \bar{b}_2 = g^{-2}h^{-6}, \bar{c}_2 = gh^4$$

form a Beauville structure, and are transposed by the automorphism $g \mapsto h, h \mapsto g$ of P .

Now let $G = H_1 \times H_2 \cong H^2$ where

$$H_1 = \langle u, g \mid u^q = g^p = 1, u^g = u^\lambda \rangle$$

and

$$H_2 = \langle v, h \mid v^q = h^p = 1, v^h = v^\lambda \rangle$$

are isomorphic copies of the non-abelian group H of order pq for a prime $q \equiv 1 \pmod{p}$, with $\lambda^p = 1 \neq \lambda$ in \mathbb{Z}_q . Thus G is a semidirect product of a normal subgroup $Q = \langle u, v \rangle \cong C_q \times C_q$ by $P = \langle g, h \rangle \cong C_p \times C_p$, with $Z(G) = Z(H_1) \times Z(H_2) = 1$. We lift the Beauville structure in P to G by defining triples

$$a_1 = u^i v^r g^2 h, b_1 = u^j v^s g^{-6} h^{-2}, c_1 = u^k v^t g^4 h$$

and

$$a_2 = u^r v^i gh^2, b_2 = u^s v^j g^{-2} h^{-6}, c_2 = u^t v^k gh^4,$$

transposed by the automorphism ζ of G which transposes g and h , and u and v . The condition that each triple should have product 1 can be written as

$$1 = u^i g^2 . u^j g^{-6} . u^k g^4 = u^i . g^2 u^j g^{-2} . g^{-4} u^k g^4 = u^{i+j\lambda^{-2}+k\lambda^4}$$

in H_1 and

$$1 = v^r h . v^s h^{-2} . v^t h = v^r . h v^s h^{-1} . h^{-1} v^t h = v^{r+s\lambda^{-1}+t\lambda},$$

in H_2 , that is,

$$i + j\lambda^{-2} + k\lambda^4 = r + s\lambda^{-1} + t\lambda = 0$$

in \mathbb{Z}_q .

All six elements in these triples have order p , since their images in P act without fixed points on $Q \setminus \{1\}$. It follows that no non-identity power of a_1, b_1 or c_1 can be conjugate in G to a power of a_2, b_2 or c_2 , since this property holds for their images in P .

By its construction, the triple a_1, b_1, c_1 maps onto a generating triple for P , so it generates a subgroup G_0 of G of order divisible by p^2 . For simplicity, let us take $i = 1, j = -\lambda^2, k = 0$ and $r = 1, s = -\lambda, t = 0$, satisfying the above conditions in \mathbb{Z}_q , so the triples are

$$a_1 = uv g^2 h, \quad b_1 = u^{-\lambda^2} v^{-\lambda} g^{-6} h^{-2}, \quad c_1 = g^4 h$$

and

$$a_2 = uv g h^2, \quad b_2 = u^{-\lambda} v^{-\lambda^2} g^{-2} h^{-6}, \quad c_2 = g h^4.$$

The projection in H_1 of the commutator $[a_1, c_1]$ is

$$[u g^2, g^4] = g^{-2} u^{-1} . g^{-4} . u g^2 . g^4 = g^{-2} u^{-1} g^2 . g^{-6} u g^6 = u^{-\lambda^2 + \lambda^6},$$

and its projection in H_2 is

$$[v h, h] = h^{-1} v^{-1} . h^{-1} . v h . h = h^{-1} v^{-1} h . h^{-2} v h^2 = v^{-\lambda + \lambda^2},$$

so G_0 contains the element

$$[a_1, c_1] = u^{-\lambda^2 + \lambda^6} v^{-\lambda + \lambda^2}$$

of Q . Conjugating this by a_1 , we see that G_0 also contains the element

$$u^{-\lambda^4 + \lambda^8} v^{-\lambda^2 + \lambda^3}$$

of Q . These two elements generate Q since the determinant

$$(-\lambda^2 + \lambda^6)(-\lambda^2 + \lambda^3) - (-\lambda + \lambda^2)(-\lambda^4 + \lambda^8) = -\lambda^4(\lambda - 1)^2(1 - \lambda^4)$$

is non-zero in \mathbb{Z}_q . Thus $G_0 = G$, so the triple a_1, b_1, c_1 generates G , and hence so does a_2, b_2, c_2 . These triples therefore form a Beauville structure in G . Since $Z(G) = 1$ the corresponding Beauville surface \mathcal{S} has only the identity inner automorphism, that is, $I = 1$. As shown in Example 4.3, the Beauville surface $\overline{\mathcal{S}}$ corresponding to P cannot have a direct outer automorphism of order 2, and it is easy to check that an automorphism of P inducing a 3-cycle on $\overline{a}_1, \overline{b}_1, \overline{c}_1$ does not leave the triple $\overline{a}_2, \overline{b}_2, \overline{c}_2$ invariant; thus P has no direct outer automorphisms, and hence the same applies to \mathcal{S} , giving $A^0 = 1$. However, \mathcal{S} has an indirect automorphism of order 2 induced by the automorphism ζ of G transposing the two triples a_i, b_i, c_i , so $A \cong C_2$.

6.4 Realising abelian groups

Using Example 6.3, we will show that every finite abelian group can arise as the inner automorphism group of a Beauville surface admitting indirect automorphisms. First we need the following lemma:

Lemma 6.2 *Let G_0, G_1, \dots, G_k be mutually orthogonal finite groups. For each $j = 0, \dots, k$ let G_j have generating triples (a_{1j}, b_{1j}, c_{1j}) and (a_{2j}, b_{2j}, c_{2j}) such that*

- (a_{10}, b_{10}, c_{10}) and (a_{20}, b_{20}, c_{20}) form a Beauville structure for G_0 , and
- each (a_{1j}, b_{1j}, c_{1j}) has type (l_j, m_j, n_j) , where l_j divides l_0 , m_j divides m_0 and n_j divides n_0 for each j .

Then the elements $a_1 = (a_{10}, \dots, a_{1k}), \dots, c_2 = (c_{20}, \dots, c_{2k})$ form a Beauville structure for the group $G = G_0 \times \dots \times G_k$.

(Note that for $j = 1, \dots, k$ we do not require the two triples in G_j to form a Beauville structure. Indeed, in some cases we can (and will) take (a_{1j}, b_{1j}, c_{1j}) and (a_{2j}, b_{2j}, c_{2j}) to be the same triple.)

Proof. If some power a_1^r of a_1 is conjugate in G to a power a_2^s of a_2 , then the same applies to their projections a_{10}^r and a_{20}^s in G_0 , so $a_{10}^r = 1$ since the two triples in G_0 form a Beauville structure. Thus r is divisible by l_0 , and hence by l_j for every j , so $a_{1j}^r = 1$ and hence $a_1^r = 1$. A similar argument applies to any other pair chosen from the triple (a_1, b_1, c_1) and the triple (a_2, b_2, c_2) . The rest of the proof follows that for Lemma 5.3. \square

Given an abelian group H , the *generalised dihedral group* $\text{Dih } H$ is the semidirect product of H by a complement C_2 inverting H by conjugation. The next result is an analogue of Theorem 5.7:

Theorem 6.3 *Each finite generalised dihedral group is isomorphic to $\text{Aut } \mathcal{S}$ for some Beauville surface \mathcal{S} with an indirect automorphism.*

Proof. Let G_0 be the group of order p^2q^2 denoted by G in Example 6.3, with a Beauville structure in which two triples (a_{10}, b_{10}, c_{10}) and (a_{20}, b_{20}, c_{20}) of type (p, p, p) are transposed by an automorphism of G_0 , for some prime $p \geq 13$ (for instance, we could take $p = 13$ and $q = 53$). Given any finite abelian group $H \cong C_{m_1} \times \dots \times C_{m_k}$, we can use Proposition 5.1 and Lemma 4.1 to choose mutually orthogonal groups G_j ($j = 1, \dots, k$) of type $SL_d(q)$, so that each G_j is a smooth quotient of $\Delta(2, 3, p)$, and hence of $\Delta(p, p, p)$, with $Z(G_j) \cong C_{m_j}$. In Lemma 6.2 we take $(a_{1j}, b_{1j}, c_{1j}) = (a_{2j}, b_{2j}, c_{2j})$ to be the corresponding generating triple of G_j of type (p, p, p) for $j = 1, \dots, k$. By Lemma 6.2 the elements $a_1 = (a_{10}, \dots, a_{1k}), \dots, c_2 = (c_{20}, \dots, c_{2k})$ form a Beauville structure for $G := G_0 \times G_1 \times \dots \times G_k$. Since $Z(G_0) = 1$ the corresponding Beauville surface \mathcal{S} has inner automorphism group $I \cong Z(G) \cong H$. The two triples (a_1, b_1, c_1) and (a_2, b_2, c_2) are transposed by an outer automorphism of G which acts as in Example 6.3 on G_0 , and as the identity on G_j for each $j = 1, \dots, k$, thus inducing an indirect automorphism τ of \mathcal{S} .

It is shown in Example 6.3 that the Beauville surface constructed there from G_0 has no direct outer automorphisms, so the same applies to \mathcal{S} . Thus $\text{Aut } \mathcal{S}$ is a semidirect product of I and a group $\langle \tau \rangle \cong C_2$. Now τ acts by conjugation on I as $(\alpha_1, \alpha_2) \mapsto (\alpha_2, \alpha_1)$, and since we identify (α_1, α_2) with the element $\alpha_1 \alpha_2^{-1} \in Z(G)$ it follows that τ acts on I by inverting each element, so $\text{Aut } \mathcal{S} \cong \text{Dih } I \cong \text{Dih } H$. \square

In Theorem 6.3, the order of the outer automorphism group $\text{Out } \mathcal{S}$ attains its lower bound among all Beauville surfaces with indirect automorphisms, namely $|\text{Out } \mathcal{S}| = 2$. We will now construct examples in which it attains its upper bound of 72.

Example 6.4 Let $G = G_1 \times G_2$, where $G_1 = L_2(5^2)$ and $G_2 = L_2(3^3)$. Our aim is to construct a Beauville structure of type $(13, 13, 13; 13, 13, 13)$ for G , even though it follows easily from Sylow's Theorems that neither G_1 nor G_2 can have a structure of this type, since each has cyclic Sylow 13-subgroups.

It can be seen from their character tables and lists of maximal subgroups in [8], or alternatively deduced from results of Macbeath [26], that each G_j is a smooth quotient of $\Delta(2, 3, 13)$, and hence by Lemma 4.1 of $\Delta(13, 13, 13)$, giving a generating triple (a_{1j}, b_{1j}, c_{1j}) of type $(13, 13, 13)$ with all three generators in the same conjugacy class. In each case, this can be chosen to be any of the six conjugacy classes of elements of order 13 in G_j . Note that the elements of each such class are conjugate to their inverses, but to no other proper powers of themselves.

In G_1 we take a_{21}, b_{21} and c_{21} to be the images of a_{11}, b_{11} and c_{11} under the automorphism of order 2 induced by the Frobenius automorphism $z \mapsto z^5$ of the underlying field \mathbb{F}_{5^2} . These elements lie in a conjugacy class consisting of the 5th powers of those in the class containing a_{11}, b_{11} and c_{11} , so an element of the first triple is conjugate to the r th power of an element of the second triple if and only if $r \equiv \pm 5 \pmod{13}$. In G_2 we take $a_{22} = a_{12}$, $b_{22} = b_{12}$ and $c_{22} = c_{12}$, so in this case an element of the first triple is conjugate to the r th power of an element of the second triple if and only if $r \equiv \pm 1 \pmod{13}$. It follows that in G , no element of the triple $a_1 = (a_{11}, a_{12}), b_1 = (b_{11}, b_{12}), c_1 = (c_{11}, c_{12})$ can be conjugate to a power of an element of the triple $a_2 = (a_{21}, a_{22}), b_2 = (b_{21}, b_{22}), c_2 = (c_{21}, c_{22})$.

Since G_1 and G_2 are mutually orthogonal (as non-isomorphic simple groups), each of these triples generates G , so they form a Beauville structure of type $(13, 13, 13; 13, 13, 13)$ in G . The two triples are transposed by an outer automorphism ζ of G , which acts as the field automorphism on G_1 and the identity on G_2 , so by Proposition 3.3 the corresponding Beauville surface \mathcal{S} has an indirect automorphism τ . Now $\text{Inn } \mathcal{S} \cong Z(G) = 1$, and Lemma 4.1 implies that each curve \mathcal{C}_i ($i = 1, 2$) has an automorphism group S_3 commuting with G , so \mathcal{S} has direct automorphism group $\text{Aut}^0 \mathcal{S} \cong S_3 \times S_3$. The existence of τ shows that $\text{Aut } \mathcal{S}$ properly contains $\text{Aut}^0 \mathcal{S}$, so by Proposition 6.1 it is isomorphic to $S_3 \wr S_2$.

We can extend the above example by proving an analogue of Theorem 6.3, in which $|\text{Out } \mathcal{S}|$ now attains its upper bound of 72. We can again use Lemma 6.2, but instead of taking G_0 to be the group in Example 6.3, as used in the proof of Theorem 6.3, we use the group $L_2(5^2) \times L_2(3^3)$ in Example 6.4. Otherwise, the proof follows that of Theorem 6.3, with $p = 13$, except that now every group G_j ($j = 0, \dots, k$) arises as a quotient of $\Delta(2, 3, 13)$; this implies that the curves \mathcal{C}_1 and \mathcal{C}_2 each have an automorphism group S_3 .

commuting with G , so that $\text{Out}^0 \mathcal{S} \cong S_3 \times S_3$. An indirect automorphism τ , acting as the field automorphism of $L_2(5^2)$, and as the identity on $L_2(3^3)$ and the special linear groups G_1, \dots, G_k , transposes the two direct factors S_3 of $\text{Out}^0 \mathcal{S}$, so $\text{Out} \mathcal{S} \cong S_3 \wr S_2$. This proves:

Theorem 6.4 *Every finite abelian group is isomorphic to the inner automorphism group of some Beauville surface with outer automorphism group isomorphic to $S_3 \wr S_2$.* \square

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